

THURSTON TYPE THEOREM FOR SUB-HYPERBOLIC RATIONAL MAPS

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ABSTRACT. In 1980's, Thurston established a combinatorial characterization for post-critically finite rational maps. This criterion was then extended by Cui, Jiang, and Sullivan to sub-hyperbolic rational maps. The goal of this paper is to present a new but simpler proof of this result by adapting the argument in the proof of Thurston's Theorem.

1. INTRODUCTION

Let $f : S^2 \rightarrow S^2$ be an orientation-preserving branched covering map of degree $d \geq 2$. We denote by $\deg_x f$ the local degree of f at x . We will call

$$\Omega_f = \{x \in S^2 \mid \deg_f(x) \geq 2\}$$

the critical set of f and

$$P_f = \overline{\bigcup_{k \geq 1} f^k(\Omega_f)}.$$

the post-critical set. We say f is post-critically finite if P_f is a finite set. In 1980's, Thurston established a combinatorial characterization for post-critically finite rational maps. The theorem says that if the associated orbifold \mathcal{O}_f is hyperbolic, then f is combinatorially equivalent to a rational map if and only if it has no Thurston obstructions. The basic idea of the proof is as follows. Consider the Teichmüller space T_f modeled on (S^2, P_f) . Then f induces an analytic operator $\sigma_f : T_f \rightarrow T_f$. It turns out that the existence of a rational map which realizes f is equivalent to the existence of a fixed point of σ_f . The proof is then reduced to showing that σ_f is a strictly contracting map. The reader may refer to [5] for a detailed proof of this theorem.

A natural question is that to what extent, Thurston's theorem can be extended to rational maps with infinitely many post-critical points. It was proved by McMullen that having no Thurston obstruction is essentially true for any rational map with a hyperbolic orbifold — only trivial Thurston obstructions inside Siegel disks or Herman rings may occur for a rational map with a hyperbolic orbifold [7]. In 1994, Cui, Jiang, and Sullivan established a Thurston type theorem for sub-hyperbolic rational maps ([2], see also [4], [8]).

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The original proof of Cui-Jiang-Sullivan's theorem is quite involved. The goal of this paper is to give a new but simpler proof of this theorem by adapting the argument used in the proof of Thurston's theorem.

Before we present this theorem, let us introduce some definitions first. We say f is geometrically finite if P_f is an infinite set but with finitely many accumulation points. Suppose that f is geometrically finite. Then it is not difficult to see that the accumulation set of P_f consists of finitely many periodic cycles. We leave this to the reader as an exercise. Let P'_f denote the set of all the accumulation points of P_f .

Definition 1.1. Let $f : S^2 \rightarrow S^2$ be a geometrically finite branched covering map of degree $d \geq 2$. We say f is a sub-hyperbolic semi-rational branched covering if for any $a \in P'_f$ of period $p \geq 1$, there is an open neighborhood U of a , such that f is holomorphic in U , and moreover, if $\deg_a f^p = 1$, then

$$f^p(z) = a + \lambda(z - a) + o(|z - a|) \text{ for } z \in U$$

where $0 < |\lambda| < 1$ is some constant, and if $\deg_a f^p = k > 1$, then

$$f^p(z) = a + \alpha(z - a)^k + o(|z - a|^k) \text{ for } z \in U$$

where $\alpha \neq 0$ is some constant.

As in the post-critically finite case, one can define Thurston obstructions for a sub-hyperbolic semi-rational branched covering map f in a similar way. If γ is a simple closed curve in $S^2 \setminus P_f$, then the set $f^{-1}(\gamma)$ is a union of disjoint simple closed curves. If γ moves continuously, so does each component of $f^{-1}(\gamma)$. A simple closed curve γ is non-peripheral if each component of $S^2 \setminus \gamma$ contains at least two points of P_f . Consider a multi-curve

$$\Gamma = \{\gamma_1, \dots, \gamma_n\}$$

of simple, closed, disjoint, non-homotopic, and non-peripheral curves in $S^2 \setminus P_f$. We say that Γ is f -stable if for any $\gamma \in \Gamma$, every non-peripheral component of $f^{-1}(\gamma)$ is homotopic in $S^2 \setminus P_f$ to an element of Γ .

For each f -stable multi-curve Γ , define a linear transformation,

$$f_\Gamma : \mathbb{R}^\Gamma \rightarrow \mathbb{R}^\Gamma$$

as follows: let $\gamma_{i,j,\alpha}$ denote the components of $f^{-1}(\gamma_j)$ homotopic to γ_i in $S^2 \setminus P_f$ and $d_{i,j,\alpha}$ be the degree of $f|_{\gamma_{i,j,\alpha}} : \gamma_{i,j,\alpha} \rightarrow \gamma_j$. Define

$$f_\Gamma(\gamma_j) = \sum_i \left(\sum_\alpha \frac{1}{d_{i,j,\alpha}} \right) \gamma_i.$$

Since the matrix of f_Γ is non-negative, there exists a largest eigenvalue $\lambda(\Gamma, f) \in \mathbb{R}_+$. We say that a multi-curve Γ is a Thurston obstruction of f if $\lambda(\Gamma, f) \geq 1$.

Definition 1.2. Suppose f and g are two sub-hyperbolic semi-rational branched coverings. We say that they are CLH-equivalent (combinatorially

and locally holomorphically equivalent) if there exist a pair of homeomorphisms $\phi : S^2 \rightarrow S^2$ and $\psi : S^2 \rightarrow S^2$ such that

- ψ is isotopic to ϕ rel \overline{P}_f ,
- $\phi f = g\psi$,
- $\phi|_{U_f} = \psi|_{U_f}$ is holomorphic on some open set $U_f \supset P'_f$.

Now let us state the Thurston type theorem for sub-hyperbolic rational maps.

Main Theorem. *Suppose f is a sub-hyperbolic semi-rational branched covering. Then f is CLH-equivalent to a rational map R if and only if f has no Thurston obstructions. In this case, the rational map R is unique up to a Möbius conjugation of the Riemann sphere.*

Remark 1.1. There are branched covering maps of the sphere which are geometrically finite and having no Thurston obstructions but are not combinatorially equivalent to rational maps. For the construction of such maps, see [3].

The proof of the "only if" part follows from a theorem of McMullen (see Appendix B of [7]). The main task of this paper is to prove the "if" part.

The essential difference between the post-critically finite case and the sub-hyperbolic case is that in the first case, the post-critical set is a finite set and the Thurston pull back induces an analytic operator defined on a finite-dimensional Teichmüller space, while in the latter case, the post-critical set is an infinite set and therefore, the induced operator is defined on an infinite-dimensional Teichmüller space. However, we observe in this paper that, in both cases, the following bounded geometry properties are similar. This allows us to prove the latter case by adapting the argument in the proof of the first case.

In the post-critically finite case, the base point of the Teichmüller space is the Riemann sphere minus the set of finite number of post-critical points. The branched covering induces a pull-back operator on this Teichmüller space. Iterations of this operator produce a sequence of sets of finite number of points in the Riemann sphere. The bounded geometry in this case means that there is a positive constant such that any two points in any element of this sequence have spherical distance greater than or equal to this constant.

In the sub-hyperbolic case, the base point of the Teichmüller space is the Riemann sphere minus the union of finitely many points and topological disks. Iterations of the pull-back operator produce a sequence of sets of finite number of points plus finite number of disks in the Riemann sphere. The bounded geometry in this case means that there is a positive constant such that in any element of this sequence, the spherical distance between any two points, any point and any disk, or any two disks is greater than or equal to this constant;

moreover, any disk in any element of this sequence contains another round disk of radius greater than or equal to this constant.

The paper is organized as follows. In §2, we prove the Shielding Ring Lemma. The proof is elementary but it is crucial in our construction of the Teichmüller space. In §3, we construct the Teichmüller space T_f . In §4, we introduce the pull back operator $\sigma_f : T_f \rightarrow T_f$. In §5, we introduce the concept of bounded geometry. In §6, we prove that bounded geometry implies the strictly contracting property of σ_f . In §7, we prove that no Thurston obstruction implies the bounded geometry. This completes the proof of the Main Theorem.

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2. SHIELDING RING LEMMA

We say an open annulus A is attached to an open topological disk D from the outside if A and D are disjoint but ∂D is the inner boundary component of the annulus A . Then $\overline{D \cup A}$ is a larger closed disk.

Suppose that f is a sub-hyperbolic semi-rational branched covering. Let $P'_f = \{a_i\}$. The main purpose of this section is to prove the following lemma.

Lemma 2.1 (Shielding Ring Lemma). *There is a collection $\{D_i\}$ of open disks and a collection of open annuli $\{A_i\}$ such that*

- $a_i \in D_i$,
- $\overline{D_i} \cap \overline{D_j} = \emptyset$ for $i \neq j$,
- for each i , A_i is an annulus attaching D_i from the outside such that $\overline{A_i} \cap P_f = \emptyset$,
- f is holomorphic on $\overline{D_i} \cup A_i$,
- every $f(A_i)$ is contained in some D_j .

Proof. Since P'_f consists of finitely many periodic cycles, it is sufficient to find D_i and A_i for each periodic cycle.

Suppose

$$\{a_1, \dots, a_p\}$$

is a periodic cycle in P'_f such that

$$f(a_i) = a_{i+1 \pmod{p}}, \quad 1 \leq i \leq p.$$

This periodic cycle is either attracting or super-attracting. Let us assume that we are in the attracting case. That is, we can find a topological disk W containing a_1 and a holomorphic isomorphism $\phi : W \rightarrow \Delta$ such that

$\phi \circ f^p \circ \phi^{-1} : \Delta \rightarrow \Delta$ is equal to λz for some $0 < |\lambda| < 1$. The super-attracting case can be treated in a similar way by making minor changes.

For $0 < r < 1$, let $\mathbb{T}_r = \{z \mid |z| = r\}$ and $\Delta_r = \{z \mid |z| < r\}$. Let $U_r = \phi^{-1}(\Delta_r)$. Note that there are only countably many r such that

$$\bigcup_{i \geq 0} f^i(\partial U_r) \cap P_f \neq \emptyset.$$

So we can take $0 < a < 1$ such that

$$(1) \quad \bigcup_{i \geq 0} f^i(\partial U_a) \cap P_f \neq \emptyset.$$

Let $b = a + \epsilon < 1$ for some $\epsilon > 0$ small. Let

$$H = \{z \mid a < |z| < b\}.$$

From (1), it follows that by taking $\epsilon > 0$ small, we can assume

1. $\bigcup_{0 \leq i \leq p-1} f^i(\phi^{-1}(\overline{H})) \cap P_f = \emptyset$, and
2. $f^p(\phi^{-1}(\overline{H})) \subset U_a$.

Now divide the annulus H into p sub-annuli H_1, \dots, H_p as follows. Take $a = r_0 < r_1 < r_2 < \dots < r_p = b$. Let $H_i = \{z \mid r_{i-1} < |z| < r_i\}$. Let $E_i = \phi^{-1}(H_i)$. Define

$$D_1 = U_a \quad \text{and} \quad A_1 = E_1,$$

and

$$D_2 = f(\overline{U_a} \cup E_1) \quad \text{and} \quad A_2 = f(E_2).$$

For $3 \leq i \leq p$,

$$D_i = f^{i-1}(\overline{U_a} \cup \bigcup_{1 \leq j \leq i-2} \overline{E_j} \cup E_{i-1}) \quad \text{and} \quad A_i = f^{i-1}(E_i).$$

After we did for every periodic cycle in P'_f , we put those disks and annuli together to get a collection of open topological disks $\{D_i\}$ and a collection of open annuli $\{A_i\}$. By the construction, it is clear that they satisfy the properties in Lemma 2.1. This completes the proof of Lemma 2.1. \square

We call the disk D_i in Lemma 2.1 a holomorphic disk and the corresponding annulus A_i a shielding ring.

Remark 2.1. By our construction, the boundary of every D_i is a real-analytic curve.

3. THE TEICHMÜLLER SPACE T_f

Let us now fix a collection of holomorphic disks $\{D_i\}$ and a collection of shielding rings $\{A_i\}$ for f . Let

$$D_f = \bigcup_i D_i \quad \text{and} \quad P_1 = P_f \setminus D_f.$$

By taking D_i smaller, we may assume that $\#(P_1) \geq 3$. We may further assume that $\{0, 1, \infty\} \subset P_1$. Define

$$Q_f = P_1 \cup \overline{D_f} \quad \text{and} \quad X_f = \partial Q_f = P_1 \cup \partial D_f.$$

Definition 3.1. The Teichmüller space T_f is the Teichmüller space modeled on $(S^2 \setminus Q_f, X_f)$.

The Teichmüller space T_f can be constructed as the space of all the Beltrami coefficients defined on $S^2 \setminus Q_f$ module the following equivalent relation: let μ and ν be two Beltrami coefficients defined on $S^2 \setminus Q_f$ and let

$$\phi_\mu : S^2 \setminus Q_f \rightarrow S \quad \text{and} \quad \phi_\nu : S^2 \setminus Q_f \rightarrow R$$

be two quasiconformal homeomorphisms which solve the Beltrami equations given by μ and ν , respectively. we say μ and ν are equivalent to each other if there exists a holomorphic isomorphism $h : R \rightarrow S$ such that the map ϕ_μ and $h \circ \phi_\nu$ are isotopic to each other rel X_f , that is, there is a continuous family of quasiconformal homeomorphisms $g_t : S^2 \setminus Q_f \rightarrow S$, $0 \leq t \leq 1$, such that

1. $g_0 = \phi_\mu$,
2. $g_1 = h \circ \phi_\nu$,
3. $g_t(z) = \phi_\mu(z) = (h \circ \phi_\nu)(z)$ for all $0 \leq t \leq 1$ and $z \in X_f$.

Now let us give a brief description of the relative background about the Teichmüller space T_f . The reader may refer to [6] for more knowledge in this aspect.

Let μ be a Beltrami coefficient defined on $S^2 \setminus Q_f$. Let

$$\phi_\mu : S^2 \setminus Q_f \rightarrow \phi_\mu(S^2 \setminus Q_f)$$

be a quasiconformal homeomorphism which solves the Beltrami equation given by μ . Let

$$M_\mu = \left\{ \xi(z) \frac{d\bar{z}}{dz} \mid \xi(z) \text{ is measurable and } \|\xi\|_\infty < \infty \right\}$$

be the linear space of all the Beltrami differentials defined on $\phi_\mu(S^2 \setminus Q_f)$. Let

$$A_\mu = \left\{ q(z) dz^2 \mid q(z) \text{ is holomorphic and } \int_{\phi_\mu(S^2 \setminus Q_f)} |q(z)| dz \wedge d\bar{z} < \infty \right\}$$

be the linear space of all the integrable holomorphic quadratic differentials defined on $\phi_\mu(S^2 \setminus Q_f)$.

A Beltrami differential $\xi(z) \frac{d\bar{z}}{dz} \in M_\mu$ is called *infinitesimally trivial* if

$$\int_{\phi_\mu(S^2 \setminus Q_f)} \xi(z) q(z) dz \wedge d\bar{z} = 0$$

holds for all $q(z) dz^2 \in A_\mu$.

Let $N_\mu \subset M_\mu$ be the subspace of all the *infinitesimally trivial* Beltrami differentials. Then the tangent space of T_f at $[\mu]$ is isomorphic to the quotient space M_μ/N_μ .

Let μ be a Beltrami coefficient defined on $S^2 \setminus Q_f$. Let ξ be a tangent vector of T_f at $[\mu]$ which is identified with a Beltrami differential $\xi(z) \frac{d\bar{z}}{dz}$ defined on $\phi_\mu(S^2 \setminus Q_f)$.

Definition 3.2. The Teichmüller norm of the tangent vector ξ is defined to be

$$\|\xi\| = \sup \left| \int_{\phi_\mu(S^2 \setminus Q_f)} q(z) \xi(z) dz \wedge d\bar{z} \right|,$$

where the sup is taken over all $q(z) dz^2 \in A_\mu$ with $\int_{\phi_\mu(S^2 \setminus Q_f)} |q(z)| dz \wedge d\bar{z} = 1$.

Definition 3.3. Let $[\mu], [\nu] \in T_f$. The Teichmüller distance $d_T([\mu], [\nu])$ is define to be

$$\frac{1}{2} \inf \log K(\phi_{\mu'} \circ \phi_{\nu'}^{-1})$$

where $\phi_{\mu'}$ and $\phi_{\nu'}$ are quasi-conformal mappings with Beltrami coefficients μ' and ν' and the inf is taken over all μ' and ν' in the same Teichmüller classes as μ and ν , respectively.

Lemma 3.1. Let μ and ν be two Beltrami coefficients defined on $S^2 \setminus Q_f$. Then

$$d_T([\mu], [\nu]) = \inf \int_0^1 \|\tau'(t)\| dt$$

where inf is taken over all the piecewise smooth curves $\tau(t)$ in T_f such that $\tau(0) = [\mu]$ and $\tau(1) = [\nu]$.

4. THE PULL-BACK OPERATOR

As in the post-critically finite case, we may assume that f is a quasiconformal map (This is because except the finite holomorphic disks, there are only finitely many points in P_f , and therefore, the CLH-equivalent class of f must contain a quasiconformal branched covering map of the sphere). From now on, we use \mathbb{P}^1 to denote the two sphere endowed with the standard complex structure.

Remind that for a Beltrami coefficient μ defined on the sphere S^2 , the pull back of μ by f , which is denoted by $f^*(\mu)$, is defined to be

$$(2) \quad (f^*\mu)(z) = \frac{\mu_f(z) + \mu(f(z))\theta(z)}{1 + \overline{\mu_f(z)}\mu(f(z))\theta(z)}$$

where $\theta(z) = \overline{f_z}/f_z$ and $\mu_f(z) = f_{\bar{z}}/f_z$. It is important to note that if μ depends complex analytically on t , then so does $f^*(\mu)$.

Now let $\mu(z)$ be a Beltrami coefficient defined on $S^2 \setminus Q_f$. Define the Beltrami coefficient $\text{Ext}(\mu)(z)$ on S^2 by setting

$$(3) \quad \text{Ext}(\mu)(z) = \begin{cases} \mu(z) & \text{for } z \in S^2 \setminus Q_f, \\ 0 & \text{for otherwise.} \end{cases}$$

By (2), $f^*(\text{Ext}(\mu))$ is a Beltrami coefficient on the sphere S^2 . Let us simply use $f^*(\mu)$ to denote the restriction of $f^*(\text{Ext}(\mu))$ on $S^2 \setminus Q_f$.

Lemma 4.1. *The map f^* induces a complex analytic operator $\sigma_f : T_f \rightarrow T_f$.*

Proof. Suppose μ and ν are two Beltrami coefficients defined on $S^2 \setminus Q_f$ which are equivalent to each other. Let $\text{Ext}(\mu)$ and $\text{Ext}(\nu)$ be their extensions to S^2 . Let $\phi_{\text{Ext}(\mu)}$ and $\phi_{\text{Ext}(\nu)}$ be the corresponding quasiconformal homeomorphisms of the sphere which fix 0, 1, and the infinity. Let ϕ_μ and ϕ_ν denote their restrictions to $S^2 \setminus Q_f$, respectively. Since μ is equivalent to ν , we have a holomorphic isomorphism

$$h : \mathbb{P}^1 \setminus \phi_{\text{Ext}(\nu)}(Q_f) \rightarrow \mathbb{P}^1 \setminus \phi_{\text{Ext}(\mu)}(Q_f)$$

such that ϕ_μ is isotopic to $h \circ \phi_\nu$ rel X_f . Now define a homeomorphism $\text{Ext}(h) : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ by setting

$$(4) \quad \text{Ext}(h)(z) = \begin{cases} h(z) & \text{for } z \in \mathbb{P}^1 \setminus \phi_{\text{Ext}(\nu)}(Q_f), \\ \phi_{\text{Ext}(\mu)} \circ \phi_{\text{Ext}(\nu)}^{-1}(z) & \text{for otherwise.} \end{cases}$$

It is clear that $\text{Ext}(h)$ is holomorphic everywhere except those points in $\phi_{\text{Ext}(\nu)}(X_f)$. Since $\phi_{\text{Ext}(\nu)}(X_f)$ is the union of finitely many points and finitely many quasi-circles (see Remark 2.1), it follows that $\text{Ext}(h)$ is a holomorphic homeomorphism of the sphere to itself, and therefore a Möbius map. By the normalization condition, $\text{Ext}(h)$ fixes 0, 1, and ∞ also. So $\text{Ext}(h) = \text{id}$. This implies that ϕ_μ and ϕ_ν are isotopic to each other rel X_f , and in particular, $\phi_\mu = \phi_\nu$ on X_f . Since $\phi_{\text{Ext}(\mu)}$ and $\phi_{\text{Ext}(\nu)}$ are holomorphic on D_f , it follows that $\phi_{\text{Ext}(\mu)} = \phi_{\text{Ext}(\nu)}$ on Q_f and therefore are isotopic to each other rel Q_f . Since $f(Q_f) \subset Q_f$, we can therefore lift this isotopy and get a isotopy between $\phi_{f^*(\text{Ext}(\mu))}$ and $\phi_{f^*(\text{Ext}(\nu))}$ rel Q_f . It follows that $\phi_{f^*(\mu)}$ and $\phi_{f^*(\nu)}$, which are respectively the restrictions of $\phi_{f^*(\text{Ext}(\mu))}$ and $\phi_{f^*(\text{Ext}(\nu))}$ on $S^2 \setminus Q_f$, are isotopic to each other rel X_f . This implies that $[f^*(\mu)] = [f^*(\nu)]$. Let $\sigma_f([\mu]) = [f^*(\mu)]$.

Now let us show that σ_f is complex analytic. Suppose that we have a curve $\tau(t)$ in T_f such that $\tau(t)$ depends complex analytically on t when t varies in a small disk $\{t \mid |t| < \epsilon\}$. We may assume that $\epsilon > 0$ is small enough so that the following arguments are valid. Let $[\mu] = \tau(0)$. Then the map ϕ_μ induces an isometry between T_f and the Teichmüller space modeled on $(\mathbb{P}^1 \setminus \phi_\mu(Q_f), \phi_\mu(X_f))$. This isometry maps the curve $\tau(t)$ to a complex analytic curve $\theta(t)$, $|t| < \epsilon$, which passes through the origin. Since $\epsilon > 0$ is

small, by Ahlfors-Weill's formula (see Lemma 7, Chapter 5 of [6]), there is a curve of Beltrami coefficients $\eta(t)$ defined on $\mathbb{P}^1 \setminus \phi_\mu(Q_f)$ such that $[\eta(t)] = \theta(t)$ and $\eta(t)$ depends complex analytically on t when t varies in the disk $\{t \mid |t| < \epsilon\}$. Using formula (2), we can pull back $\eta(t)$ by ϕ_μ and get a curve of Beltrami coefficients $\gamma(t)$, $|t| < \epsilon$, defined on $S^2 \setminus Q_f$. It follows that $[\gamma(t)] = \tau(t)$ and $\gamma(t)$ depends complex analytically on t when t varies in the disk $\{t \mid |t| < \epsilon\}$. From (2), it follows that $\tilde{\gamma}(t)$, $|t| < \epsilon$, is also a curve of complex analytic Beltrami coefficients defined on $S^2 \setminus Q_f$. Now by Bers Embedding Theorem (see Theorem 1, Chapter 5 of [6]), the curve $\sigma_f(\tau(t)) = [\tilde{\gamma}(t)]$ is a curve in T_f which depends complex analytically on t when t varies in the disk $\{t \mid |t| < \epsilon\}$. This proves that σ_f is a complex analytic operator. \square

Once no confusion is caused, let us simply use μ to denote either $\text{Ext}(\mu)$ or μ . Let $\tilde{\mu}(z) = f^*(\mu)$.

Let $\phi_\mu, \phi_{\tilde{\mu}} : S^2 \rightarrow \mathbb{P}^1$ denote the quasiconformal homeomorphisms which fix 0, 1, and the infinity and which solve the Beltrami equations given by μ and $\tilde{\mu}$, respectively. Let

$$g = \phi_\mu \circ f \circ \phi_{\tilde{\mu}}^{-1}.$$

It is clear that g is a rational map and the following diagram commutes.

$$\begin{array}{ccc} (S^2, Q_f) & \xrightarrow{\phi_{\tilde{\mu}}} & (\mathbb{P}^1, \phi_{\tilde{\mu}}(Q_f)) \\ \downarrow f & & \downarrow g \\ (S^2, Q_f) & \xrightarrow{\phi_\mu} & (\mathbb{P}^1, \phi_\mu(Q_f)) \end{array}$$

Now suppose that ξ is a tangent vector of T_f at $\tau = [\mu]$. This means that there is a smooth curve of Beltrami coefficients $\gamma(t)$ defined on $S^2 \setminus Q_f$, such that $\gamma(0) = \mu$ and

$$(5) \quad \xi = \left. \frac{d}{dt} \right|_{t=0} \mu_{\phi_{\gamma(t)} \circ \phi_\mu^{-1}}$$

Let $d\sigma_f|_\tau$ denote the tangent map of σ_f at τ . Let $\tilde{\xi} = d\sigma_f|_\tau(\xi)$.

Lemma 4.2. *Let ξ and $\tilde{\xi}$ be as above. Then*

$$(6) \quad \tilde{\xi}(w) = \xi(g(w)) \frac{\overline{g'(w)}}{g'(w)}.$$

Proof. Note that

$$\tilde{\xi} = \left. \frac{d}{dt} \right|_{t=0} \mu_{\phi_{\gamma(t)} \circ f \circ \phi_{\tilde{\mu}}^{-1}} = \left. \frac{d}{dt} \right|_{t=0} \mu_{\phi_{\gamma(t)} \circ \phi_\mu^{-1} \circ \phi_\mu \circ f \circ \phi_{\tilde{\mu}}^{-1}} = \left. \frac{d}{dt} \right|_{t=0} \mu_{\phi_{\gamma(t)} \circ \phi_\mu^{-1} \circ g}.$$

Since g is a rational map, by (2) we have

$$\mu_{\phi_{\gamma(t)} \circ \phi_\mu^{-1} \circ g}(w) = \mu_{\phi_{\gamma(t)} \circ \phi_\mu^{-1}}(g(w)) \frac{\overline{g'(w)}}{g'(w)}$$

The Lemma then follows from (5). \square

Let $\tilde{q} = \tilde{q}(w)dw^2$ be a non-zero integrable holomorphic quadratic differential defined on $\mathbb{P}^1 \setminus \phi_{\tilde{\mu}}(Q_f)$. Define

$$(7) \quad q(z) = \sum_{g(w)=z} \frac{\tilde{q}(w)}{[g'(w)]^2}.$$

It is easy to see that $q = q(z)dz^2$ is a holomorphic quadratic differential defined on $\mathbb{P}^1 \setminus \phi_{\mu}(Q_f)$.

Proposition 4.1.

$$\int_{\mathbb{P}^1 \setminus \phi_{\mu}(Q_f)} |q(z)| dz \wedge d\bar{z} \leq \int_{\mathbb{P}^1 \setminus \phi_{\tilde{\mu}}(Q_f)} |\tilde{q}(w)| dw \wedge d\bar{w} - \int_{\cup_i \phi_{\tilde{\mu}}(A_i)} |\tilde{q}(w)| dw \wedge d\bar{w}.$$

Proof.

$$\begin{aligned} \int_{\mathbb{P}^1 \setminus \phi_{\mu}(Q_f)} |q(z)| dz \wedge d\bar{z} &= \int_{\mathbb{P}^1 \setminus \phi_{\mu}(Q_f)} \left| \sum_{g(w)=z} \frac{\tilde{q}(w)}{[g'(w)]^2} \right| dz \wedge d\bar{z} \\ &\leq \int_{(\mathbb{P}^1 \setminus \phi_{\tilde{\mu}}(Q_f)) \setminus (\cup_i \phi_{\tilde{\mu}}(A_i))} |\tilde{q}(w)| dw \wedge d\bar{w} \\ &= \int_{\mathbb{P}^1 \setminus \phi_{\tilde{\mu}}(Q_f)} |\tilde{q}(w)| dw \wedge d\bar{w} - \int_{\cup_i \phi_{\tilde{\mu}}(A_i)} |\tilde{q}(w)| dw \wedge d\bar{w} \end{aligned}$$

The first inequality comes from the fact $f(\cup A_i) \subset \cup D_i$. This completes the proof of Proposition 4.1. \square

Proposition 4.2. $\int_{\mathbb{P}^1 \setminus \phi_{\tilde{\mu}}(Q_f)} \tilde{\xi}(w) \tilde{q}(w) dw \wedge d\bar{w} = \int_{\mathbb{P}^1 \setminus \phi_{\mu}(Q_f)} \xi(z) q(z) dz \wedge d\bar{z}.$

Proof. Note that $\phi_{\tilde{\mu}}(Q_f) \subset g^{-1}(\phi_{\mu}(Q_f))$ and by (6) $\tilde{\xi}(w) = 0$ for all $w \in g^{-1}(\phi_{\mu}(Q_f)) \setminus \phi_{\tilde{\mu}}(Q_f)$. We thus have

$$\int_{\mathbb{P}^1 \setminus \phi_{\tilde{\mu}}(Q_f)} \tilde{\xi}(w) \tilde{q}(w) dw \wedge d\bar{w} = \int_{\mathbb{P}^1 \setminus g^{-1}(\phi_{\mu}(Q_f))} \tilde{\xi}(w) \tilde{q}(w) dw \wedge d\bar{w}.$$

Now Proposition 4.2 follows from (6), (7), and the fact that

$$dw \wedge d\bar{w} = \frac{dz \wedge d\bar{z}}{|g'(w)|^2}.$$

\square

As a direct consequence of Propositions 4.1 and 4.2, we have

Corollary 4.1. Let $\tau \in T_f$. Then $\|d\sigma_f|_{\tau}\| \leq 1$.

Remark 4.1. Corollary 4.1 also follows from the general fact that a complex analytic operator does not increase the Kobayashi's metric. But this particular argument we used here will be established in the latter sections to prove a strict inequality (see Corollary 6.1).

The next lemma reduces the proof of the Main Theorem to showing that the pull back operator σ_f has a unique fixed point in T_f .

Lemma 4.3. *The map f is CLH-equivalent to a unique rational map (up to Möbius conjugations) if and only if σ_f has a unique fixed point in T_f .*

Proof. If σ_f has a fixed point $[\mu]$ in T_f , then $\tilde{\mu} = f^*\mu \sim \mu$. Let $\text{Ext}(\mu)$ be the extension of μ to S^2 . Let $\phi_{\text{Ext}(\mu)}$ and $\phi_{f^*(\text{Ext}(\mu))}$ be the corresponding quasiconformal homeomorphisms which fix 0, 1, and the infinity. Let ϕ_μ and $\phi_{\tilde{\mu}}$ be their restrictions to $S^2 \setminus Q_f$, respectively. It follows that there is a conformal isomorphism

$$h : \mathbb{P}^1 \setminus \phi_\mu(Q_f) \rightarrow \mathbb{P}^1 \setminus \phi_{\tilde{\mu}}(Q_f)$$

such that $\phi_{\tilde{\mu}}$ and $h \circ \phi_\mu$ are isotopic to each other rel X_f . As in the proof of Lemma 4.1, one can show that such h is actually equal to the identity map. In fact, we can again define a homeomorphism $\text{Ext}(h) : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ by setting

$$(8) \quad \text{Ext}(h)(z) = \begin{cases} h(z) & \text{for } z \in \mathbb{P}^1 \setminus \phi_{\text{Ext}(\mu)}(Q_f), \\ \phi_{f^*(\text{Ext}(\mu))} \circ \phi_{\text{Ext}(\mu)}^{-1}(z) & \text{for otherwise.} \end{cases}$$

It is clear that $\text{Ext}(h)$ is holomorphic everywhere except those points in $\phi_{\text{Ext}(\mu)}(X_f)$. Since $\phi_{\text{Ext}(\mu)}(X_f)$ is the union of finitely many points and finitely many quasi-circles (see Remark 2.1), it follows that $\text{Ext}(h)$ is a holomorphic homeomorphism of the sphere to itself, and therefore a Möbius map. By the normalization condition, $\text{Ext}(h)$ fixes 0, 1, and ∞ also. So $\text{Ext}(h) = \text{id}$. This implies that ϕ_μ and $\phi_{\tilde{\mu}}$ are isotopic to each other rel X_f . It follows that $\phi_{\text{Ext}(\mu)}$ and $\phi_{f^*(\text{Ext}(\mu))}$ are isotopic to each other rel Q_f . Note that when restricted to D_f , $\phi_{\text{Ext}(\mu)}$ and $\phi_{f^*(\text{Ext}(\mu))}$ are analytic and equal to each other. This implies that f is CLH-equivalent to the rational map $g = \phi_{\text{Ext}(\mu)} \circ f \circ \phi_{f^*(\text{Ext}(\mu))}^{-1}$.

If f is CLH-equivalent to g , then we have a Beltrami coefficient μ defined on $S^2 \setminus Q_f$ such that $g = \phi_{\text{Ext}(\mu)} \circ f \circ \phi_{f^*(\text{Ext}(\mu))}^{-1}$ and moreover, $\phi_{\text{Ext}(\mu)}$ and $\phi_{f^*(\text{Ext}(\mu))}$ are isotopic to each other rel Q_f . This implies that ϕ_μ and $\phi_{\tilde{\mu}}$ are isotopic to each other rel X_f . It follows that $[f^*(\mu)] = [\mu]$ and thus $\sigma_f([\mu]) = [\mu]$.

It is clear that the fixed point $[\mu]$ is unique is equivalent to say that g is unique up to Möbius conjugations. \square

5. BOUNDED GEOMETRY

Let $d(X, Y)$ denote the spherical distance between two subsets of the sphere. Recall that

$$D_f = \cup_i D_i, \quad P_1 = P_f \setminus D_f, \quad \text{and } P'_f = \{a_i\}.$$

Definition 5.1. Let $b > 0$ be a constant. Let $T_{f,b} \subset T_f$ be the subspace such that for every $[\mu] \in T_{f,b}$, the following conditions hold,

(1) for all $z_i \neq z_{i'} \in P_1$,

$$d(\phi_\mu(z_i), \phi_\mu(z_{i'})) \geq b;$$

(2) for all $z_j \in P_1$ and all D_i ,

$$d(\phi_\mu(z_j), \phi_\mu(D_i)) \geq b;$$

(3) for all $D_i \neq D_{i'}$,

$$d(\phi_\mu(D_i), \phi_\mu(D_{i'})) \geq b;$$

(4) for every D_i , $\phi_\mu(D_i)$ contains a round disk of radius b centered at $\phi_\mu(a_i)$,

where $\phi_\mu : S^2 \rightarrow \mathbb{P}^1$ is the quasiconformal homeomorphism which fixes 0, 1, and the infinity, and which solves the Beltrami equation given by $\text{Ext}(\mu)$.

Let $K > 1$. Then the family of all the K -quasiconformal homeomorphisms of the sphere to itself, which fix 0, 1, and the infinity, is compact. We thus have

Lemma 5.1. *Let $K > 1$. Then for every $\delta > 0$, there is an $\epsilon > 0$ depending only on K and δ such that for every two points $x, y \in \mathbb{P}^1$ with $d(x, y) > \delta$, and every K -quasiconformal homeomorphism $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ which fixes 0, 1, and the infinity, we have $d(\phi(x), \phi(y)) > \epsilon$.*

By Definitions 3.3 and 5.1, and Lemma 5.1, we have

Lemma 5.2. *Let $b, D > 0$. Then there is a $b' > 0$ depending only on b and D such that for any two Beltrami coefficients μ and ν defined on $S^2 \setminus Q_f$, if $d_T([\mu], [\nu]) < D$ and $\mu \in T_{f,b}$, then $\nu \in T_{f,b'}$.*

Definition 5.2. Let Z be a subset of S^2 with $\#(Z) \geq 4$. Let $[\mu] \in T_f$ and $\gamma \subset S^2 \setminus Z$ be a simple closed and non-peripheral curve. We use $\|\gamma\|_{\mu,Z}$ to denote the hyperbolic length of the unique simple closed geodesic ξ which is homotopic to $\phi_\mu(\gamma)$ in the hyperbolic Riemann surface $\mathbb{P}^1 \setminus \phi_\mu(Z)$. We say γ is a (μ, Z) -simple closed geodesic if $\phi_\mu(\gamma)$ is a simple closed geodesic in $\mathbb{P}^1 \setminus \phi_\mu(Z)$.

For each holomorphic disk D_i , fix a point b_i on the boundary ∂D_i . Set

$$E = P_1 \cup \cup_i \{a_i, b_i\}.$$

Note that P_1 contains 0, 1, and the infinity by assumption. Since $P_1 \subset E$ and ϕ_μ fixes 0, 1, and the infinity, it follows that E and $\phi_\mu(E)$ contain 0, 1, and the infinity also.

Lemma 5.3. *Let $a > 0$. Then there is a $b > 0$ depending only on a such that for every Beltrami coefficient μ defined on $S^2 \setminus Q_f$ with $\mu(z) = 0$ on $\cup_i A_i$, if every (μ, E) -simple closed geodesic $\gamma \subset S^2 \setminus Q_f$ has hyperbolic length not less than a , then $\mu \in T_{f,b}$.*

Proof. Note that $\#(\phi_\mu(E)) = \#(E)$ is finite. Since $\phi_\mu(E)$ contains 0, 1, and the infinity, it follows that the spherical distance between any two points in $\phi_\mu(E)$ has a positive lower bound which depends only on a and $\#(E)$. Since ϕ_μ is holomorphic in every topological disk $\overline{D_i} \cup A_i$ and since $\phi_\mu(\overline{D_i})$ contains $\phi_\mu(a_i)$ and $\phi_\mu(b_i)$, it follows from Koebe's distortion theorem that every $\phi_\mu(D_i)$ contains a round disk centered at $\phi_\mu(a_i)$, the radius of which has a positive lower bound depending only on a . Since $\{0, 1, \infty\} \notin \phi_\mu(\overline{D_i} \cup A_i)$, it follows that the diameter of each component of $\mathbb{P}^1 \setminus \phi_\mu(A_i)$ has a positive lower bound depending only on a . Since ϕ_μ is analytic on every A_i , we have

$$\text{mod}(\phi_\mu(A_i)) = \text{mod}(A_i).$$

It follows that every $\phi_\mu(A_i)$ has definite thickness which depends only on a . All of these implies that there is a constant $b > 0$ depending only on a such that the four conditions in Definiton 5.1 hold. The proof of the lemma is completed. \square

The next lemma is a direct consequence of Proposition 6.1 and Theorem 6.3 of [5].

Lemma 5.4. *Let X be a hyperbolic Riemann surface and $\gamma \subset X$ be a simple closed geodesic with hyperbolic length l . Then there exists a topological annulus $A \subset X$ such that*

1. γ is the core curve of A ,
2. $\frac{\pi}{2l} - 1 < \text{mod}(A) < \frac{\pi}{2l}$.

From the modulus inequality of Teichmüller extremal problem (For instance, see Chapter III of [1]), we have

Lemma 5.5. *Let $T \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Let $H \subset \mathbb{P}^1$ be an annulus which separates $\{0, 1\}$ and $\{T, \infty\}$. Then*

$$\text{mod}(H) \leq \frac{1}{2\pi} \log 16(|T| + 1).$$

Lemma 5.6. *There exists an $\eta > 0$ such that for any Beltrami coefficient μ defined on $S^2 \setminus Q_f$ with $\mu(z) = 0$ on $\cup_i A_i$ and any (μ, E) -simple closed geodesic $\gamma \subset S^2 \setminus E$ with $\|\gamma\|_{\mu, E} < \eta$, we have $\gamma \subset S^2 \setminus Q_f$. Moreover, for any $\epsilon > 0$, there is a $\delta > 0$ such that*

$$(9) \quad \|\gamma\|_{\mu, E} > (1 - \epsilon)\|\gamma\|_{\mu, Q_f}$$

provided that $\|\gamma\|_{\mu, E} < \delta$.

Proof. Let $\gamma \subset S^2 \setminus E$ be a (μ, E) -simple closed geodesic. By Lemma 5.4, there is an annulus $A \subset \mathbb{P}^1 \setminus \phi_\mu(E)$ such that $\phi_\mu(\gamma)$ is the core curve of A and

$$(10) \quad \frac{\pi}{2\|\gamma\|_{\mu, E}} - 1 < \text{mod}(A) < \frac{\pi}{2\|\gamma\|_{\mu, E}}.$$

We may assume that A separates 0 and the infinity. Let K_1 and K_2 be the two components of $\mathbb{P}^1 \setminus A$ such that $0 \in K_1$ and $\infty \in K_2$. Let

$$r = \max\{|z| \mid z \in K_1\} \quad \text{and} \quad R = \min\{|z| \mid z \in K_2\}.$$

By Lemma 5.5, when $\|\gamma\|_{\mu,E}$ is small, R/r is large. Consider the round annulus

$$H = \{z \mid r < |z| < R\}.$$

It follows that $H \subset A$ and that the core curve of H is in the same homotopic class as γ . By Lemma 5.5 and (10), it follows that there is a uniform constant $0 < C < \infty$ such that

$$(11) \quad \text{mod}(H) \geq \text{mod}(A) - C$$

holds provided that $\|\gamma\|_{\mu,E}$ is small. Note that every pair $\{\phi_\mu(a_i), \phi_\mu(b_i)\}$ is contained either in $\{z \mid |z| < r\}$ or in $\{z \mid |z| > R\}$. Since ϕ_μ is holomorphic in $\overline{D_i} \cup A_i$ and $\{\phi_\mu(a_i), \phi_\mu(b_i)\} \subset \overline{D_i}$, it follows from Koebe's distortion theorem that there is an $1 < M < \infty$, which depends only on $\{D_i\}$ and $\{A_i\}$, such that every $\phi_\mu(\overline{D_i})$ is contained either in $\{z \mid |z| < Mr\}$ or in $\{z \mid |z| > R/M\}$. By (10) and (11), we have

$$R/M > Mr$$

provided that $\|\gamma\|_{\mu,E}$ is small enough. All of these implies that the annulus

$$H_M = \{z \mid Mr < |z| < R/M\}$$

is contained in $\mathbb{P}^1 \setminus \phi_\mu(Q_f)$ provided that $\|\gamma\|_{\mu,E}$ is small enough.

Now the first assertion of the lemma follows if we can show that

$$\phi_\mu(\gamma) \subset H_M$$

provided that $\|\gamma\|_{\mu,E}$ is small enough. Suppose this were not true. Then there are two cases. In the first case, there exist two points z and z' such that

1. $z \in K_2$ with $|z| = R$,
2. $|z'| = R/M$,
3. $\phi_\mu(\gamma)$ separates $\{0, z'\}$ and $\{z, \infty\}$.

In the second case, there exist two points z and z' such that

1. $|z| = Mr$,
2. $z' \in K_1$ and $|z'| = r$.
3. $\phi_\mu(\gamma)$ separates $\{0, z'\}$ and $\{z, \infty\}$.

Suppose we are in the first case. Note that the curve $\phi_\mu(\gamma)$ separates A into two sub-annuli such that the modulus of each of them is equal to $\text{mod}(A)/2$. But on the other hand, the outer one separates $\{0, z'\}$ and $\{z, \infty\}$, and thus by Lemma 5.5, its modulus has an upper bound depending only on M . By (10) this is impossible when $\|\gamma\|_{\mu,E}$ is small enough. The same argument can be used to get a contradiction in the second case. This proves the first assertion of the Lemma.

Now let us prove the second assertion. Let l denote the hyperbolic length of the core curve of H_M with respect to the hyperbolic metric of H_M . Since $H_M \subset \mathbb{P}^1 \setminus \phi_\mu(Q_f)$ when $\|\gamma\|_{\mu,E}$ is small enough, it follows that $l > \|\gamma\|_{\mu,Q_f}$. Thus we have

$$\text{mod}(H_M) = \frac{\pi}{2l} < \frac{\pi}{2\|\gamma\|_{\mu,Q_f}}.$$

From (10) and (11), there is a constant $0 < C' < \infty$ such that

$$\text{mod}(H_M) \geq \frac{\pi}{2\|\gamma\|_{\mu,E}} - C'$$

holds provided that $\|\gamma\|_{\mu,E}$ is small enough. Thus we have

$$\frac{\pi}{2\|\gamma\|_{\mu,Q_f}} \leq \frac{\pi}{2\|\gamma\|_{\mu,E}} \leq \frac{\pi}{2\|\gamma\|_{\mu,Q_f}} + C'.$$

The second assertion follows. \square

6. FROM BOUNDED GEOMETRY TO STRICTLY CONTRACTING

The main purpose of this section is to prove that bounded geometry implies the strict contracting property of the operator $\sigma_f : T_f \rightarrow T_f$. Let us first prove a technical lemma.

Lemma 6.1. *Let $H = \{z \mid 1 < |z| < R\}$ be an annulus. Let $F_n(w)$ be a sequence of integrable and holomorphic functions defined on H such that*

$$(12) \quad \int_H |F_n(w)| dw \wedge d\bar{w} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Then for any $1 < r < R$,

$$\int_{|w|=r} |F_n(w)| |dw| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof. Let $1 < r < R$ be fixed. Take $\delta > 0$ such that $1 + \delta < r < R - \delta$. Let

$$C(r, \delta) = \min\{r - 1 - \delta, R - \delta - r\}.$$

It follows that $C(r, \delta) > 0$. For any $\epsilon > 0$, by (12), there is an N such that for every $n > N$, there exist $1 < R_1 < 1 + \delta$ and $R - \delta < R_2 < R$, such that

$$\int_{|z|=R_1} |F_n(z)| |dz| < \epsilon$$

and

$$\int_{|z|=R_2} |F_n(z)| |dz| < \epsilon.$$

For $|w| = r$, by Cauchy formula, we have

$$|F_n(w)| \leq \left| \frac{1}{2\pi i} \int_{\mathbb{T}_{R_1} \cup \mathbb{T}_{R_2}} \frac{F_n(z)}{z - w} dz \right|.$$

Note that $|z - w| \geq C(r, \delta)$ for $|w| = r$ and $z \in \mathbb{T}_{R_1} \cup \mathbb{T}_{R_2}$. This implies that

$$|F_n(w)| \leq \frac{\epsilon}{\pi C(r, \delta)}$$

holds for all $|w| = r$ and $n > N$. It follows that for all $n > N$,

$$\int_{|w|=r} |F_n(w)| |dw| \leq \frac{2r\epsilon}{C(r, \delta)}.$$

The Lemma follows. \square

For a Beltrami coefficient μ defined on $S^2 \setminus Q_f$, we use $\tilde{\mu}$ to denote $f^*(\mu)$.

Lemma 6.2. *Let $b > 0$. Then there is a constant $0 < a < 1$ depending only on b such that if both $[\mu]$ and $[\tilde{\mu}]$ belong to $T_{f,b}$, then*

$$\int_{\cup \phi_{\tilde{\mu}}(A_i)} |\tilde{q}(w)| dw \wedge d\bar{w} \geq a$$

where $\tilde{q}(w)dw^2$ is any integrable holomorphic quadratic differential defined on $\mathbb{P}^1 \setminus \phi_{\tilde{\mu}}(Q_f)$ with

$$\int_{\mathbb{P}^1 \setminus \phi_{\tilde{\mu}}(Q_f)} |\tilde{q}(w)| dw \wedge d\bar{w} = 1.$$

Proof. Let us prove it by contradiction. By using a Möbius transformation which fixes 0 and 1, and maps $\phi_{\tilde{\mu}}(a_1)$ to the infinity, we may assume that $\infty \in D_1$. Since $\tilde{\mu} \in T_{f,b}$, such Möbius transformation lies in a compact family and therefore the assumption does not affect the validity of the proof.

Now let us suppose that there exist a sequence of pairs $(\tilde{\mu}_n, \mu_n)$ in $T_{f,b}$ and a sequence of holomorphic quadratic differentials \tilde{q}_n over $\mathbb{P}^1 \setminus \phi_{\tilde{\mu}_n}(Q_f)$ such that

$$(13) \quad \int_{\mathbb{P}^1 \setminus \phi_{\tilde{\mu}_n}(Q_f)} |\tilde{q}_n(w)| dw \wedge d\bar{w} = 1,$$

and

$$(14) \quad \int_{\cup \phi_{\tilde{\mu}_n}(A_i)} |\tilde{q}_n(w)| dw \wedge d\bar{w} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By Lemma 2.1 $f(\cup_i \overline{A_i}) \subset \cup_i D_i$ and f is holomorphic in $\overline{D_i} \cup A_i$. This, together with the fact that ϕ_{μ_n} is holomorphic on $\cup_i D_i$, implies that $\phi_{\tilde{\mu}_n}$ is holomorphic and thus univalent on $\cup_i (\overline{D_i} \cup A_i)$.

Note that every ring A_i is holomorphically isomorphic to some annulus

$$H_i = \{z \mid 1 < |z| < R_i\}.$$

Let $\Phi_i : H_i \rightarrow A_i$ be a holomorphic isomorphism and let \mathbb{T}_r denote the circle $\{z \mid |z| = r\}$. We claim that for every $1 < r < R_i$,

$$(15) \quad \int_{\phi_{\tilde{\mu}_n}(\Phi_i(\mathbb{T}_r))} |\tilde{q}_n(w)| |dw| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

In fact, from (14), we have

$$(16) \quad \int_{H_i} |\tilde{q}_n((\phi_{\tilde{\mu}_n} \circ \Phi_i)(z))| |(\phi_{\tilde{\mu}_n} \circ \Phi_i)'(z)|^2 dz \wedge d\bar{z} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By Lemma 6.1, we have

$$\int_{\mathbb{T}_r} |\tilde{q}_n((\phi_{\tilde{\mu}_n} \circ \Phi_i)(z))| |(\phi_{\tilde{\mu}_n} \circ \Phi_i)'(z)|^2 |dz| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since $\phi_{\tilde{\mu}_n} \circ \Phi_i$ is univalent on H_i , it follows from Koebe's 1/4-theorem that for every $1 < r < R_i$, there is a $C > 1$ depending only on r, R_i , and b such that

$$(17) \quad 1/C < |(\phi_{\tilde{\mu}_n} \circ \Phi_i)'(z)| < C$$

holds for all $z \in \mathbb{T}_r$. We thus have

$$\int_{\mathbb{T}_r} |\tilde{q}_n((\phi_{\tilde{\mu}_n} \circ \Phi_i)(z))| |(\phi_{\tilde{\mu}_n} \circ \Phi_i)'(z)| |dz| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This implies (15) and the claim has been proved. Now for every A_i , take an arbitrary $1 < r_i < R_i$ and let

$$(18) \quad \gamma_{i,n} = (\phi_{\tilde{\mu}_n} \circ \Phi_i)(\mathbb{T}_{r_i}).$$

For every n , Let R_n denote the component of $\mathbb{P}^1 \setminus \cup_i \gamma_{i,n}$ such that

$$\partial R_n = \cup_i \gamma_{i,n}.$$

Recall that $P_1 = \{z_j\}$ and $P'_f = \{a_i\}$ are both finite sets and each $\tilde{q}_n = \tilde{q}_n(w)dw^2$ has at most simple poles at the points in $\{\phi_{\tilde{\mu}_n}(z_j)\}$. This implies that one can write

$$(19) \quad \tilde{q}_n(w) = \sum_j \frac{b_{j,n}}{w - \phi_{\tilde{\mu}_n}(z_j)} + g_n(w)$$

where $g_n(w)$ is a holomorphic function on $\mathbb{P}^1 \setminus \phi_{\tilde{\mu}_n}(\overline{D_f})$.

Since $\tilde{\mu}_n \in T_{f,b}$, it follows by taking a subsequence if necessary, that we can assume that for every a_i , the sequence

$$a_{i,n} = \phi_{\tilde{\mu}_n}(a_i)$$

converges to a point e_i with respect to the spherical distance as n goes to ∞ . Since $\phi_{\tilde{\mu}_n}$ is holomorphic in $\overline{D_i} \cup A_i$, similarly, we can assume that for every D_i , the sequence

$$D_{i,n} = \phi_{\tilde{\mu}_n}(D_i)$$

converges to a topological disk E_i with respect to the Hausdorff metric. It follows that each E_i contains a round disk of radius b centered at e_i . Note that by taking each A_i thinner, we may assume that $\phi_{\tilde{\mu}_n}$ is univalent in a

larger disk containing $\overline{D_i \cup A_i}$ in its interior. So by taking a subsequence if necessary, we can also assume that

$$A_{i,n} = \phi_{\tilde{\mu}_n}(A_i)$$

converges to a topological annulus B_i with respect to the Hausdorff metric. It is clear that

$$\text{mod}(B_i) = \text{mod}(A_i).$$

Note that $\gamma_{i,n} = (\phi_{\tilde{\mu}_n} \circ \Phi_i)(\mathbb{T}_{r_i})$. Since $(\phi_{\tilde{\mu}_n} \circ \Phi_i)$ maps H_i univalently into $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and since $\tilde{\mu}_n \in T_{f,b}$, it follows again by taking a subsequence if necessary, that we may assume that $\phi_{\tilde{\mu}_n} \circ \Phi_i$ converges to some univalent function Λ_i defined on H_i , and moreover,

$$(20) \quad (\phi_{\tilde{\mu}_n} \circ \Phi_i)(z) \rightarrow \Lambda_i(z) \text{ uniformly in any compact set of } H_i.$$

Let

$$\gamma_i = \Lambda_i(\mathbb{T}_{r_i}).$$

It is not difficult to see that every γ_i is a real analytic and simple closed curve which is homotopic to the core curve of B_i .

Again by taking a subsequence if necessary, we may assume that as $n \rightarrow \infty$, for every $z_j \in P_1$,

$$w_{j,n} = \phi_{\tilde{\mu}_n}(z_j)$$

converges to some w_j in the spherical distance. It is important to note that the objects in $\{E_i\}$ and $\{w_j\}$ still satisfy the bounded geometry properties in Definition 5.1. Let

$$\mathcal{R} = \mathbb{P}^1 \setminus (\cup_i \overline{E_i} \cup \{w_j\}).$$

Since $g_n(w)$ is a holomorphic function on $\mathbb{P}^1 \setminus \phi_{\tilde{\mu}_n}(Q_f)$, it follows that for any compact set $W \subset \mathcal{R}$, the function $g_n(w)$ is defined on W provided n is large enough. Moreover, from (18), for any such compact set W , we can always take r_i close to 1 or R_i such that

$$W \subset R_n.$$

For any $w \in W$, from (19) and Cauchy formula, we have

$$\begin{aligned} g_n(w) &= \frac{1}{2\pi i} \int_{\cup_i \gamma_{i,n}} \frac{g_n(\xi)}{\xi - w} d\xi \\ &= \frac{1}{2\pi i} \int_{\cup_i \gamma_{i,n}} \frac{\tilde{q}_n(\xi)}{\xi - w} d\xi - \frac{1}{2\pi i} \sum_j \int_{\cup_i \gamma_{i,n}} \frac{b_{j,n}}{(\xi - w_{j,n})(\xi - w)} d\xi \end{aligned}$$

Note that by assumption $\infty \in D_1$ and hence $\infty \notin R_n$. It follows that

$$\frac{b_{j,n}}{(\xi - w_{j,n})(\xi - w)}$$

is holomorphic in R_n and the residues at the two simple poles are equal to each other. It follows that its integral along $\cup_i \gamma_{i,n}$ is zero. We thus have

$$g_n(w) = \frac{1}{2\pi i} \int_{\cup_i \gamma_{i,n}} \frac{\tilde{q}_n(\xi)}{\xi - w} d\xi.$$

By (15) and the fact that $d(W, \cup_i \gamma_{i,n}) > 0$, it follows that $g_n(w) \rightarrow 0$ uniformly in W as $n \rightarrow \infty$. In particular, since $\cup_i \gamma_{i,n}$ is a compact subset of \mathcal{R} , it follows that $g_n(w) \rightarrow 0$ uniformly for $w \in \cup_i \gamma_{i,n}$. This, together with (15) and (19), implies

$$(21) \quad \int_{\cup_i \gamma_{i,n}} \left| \sum_j \frac{b_{j,n}}{w - w_{j,n}} \right| |dw| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

We claim that $b_{j,n} \rightarrow 0$ as $n \rightarrow \infty$ for each j . Let us prove the claim by contradiction. Let $\beta_n = \max_j \{|b_{j,n}|\}$. By taking a subsequence we may assume that there is an $\epsilon > 0$ such that $\beta_n \geq \epsilon$ for all $n \geq 0$. Let

$$h_{j,n} = b_{j,n} / \beta_n.$$

Then $\max_j \{|h_{j,n}|\} = 1$. By (21), we have

$$(22) \quad \int_{\cup_i \gamma_{i,n}} \left| \sum_j \frac{h_{j,n}}{w - w_{j,n}} \right| |dw| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By taking a convergent subsequence again, we may assume that every $h_{j,n}$ converges to a number h_j as n goes to infinity. We thus have

$$(23) \quad \max_j \{|h_j|\} = 1.$$

From (20) and (22), we have

$$\int_{\cup_i \gamma_i} \left| \sum_j \frac{h_j}{w - w_j} \right| |dw| = 0.$$

This implies that

$$\sum_j \frac{h_j}{w - w_j} = 0 \text{ for all } w \in \cup_i \gamma_i \text{ and thus equal to zero everywhere.}$$

Since all w_j are distinct with each other, it follows by computing the residue at each w_j that all h_j are equal to zero. This contradicts with (23) and the claim has been proved.

Since $g_n(z) \rightarrow 0$ uniformly on any compact set of \mathcal{R} and $b_{j,n} \rightarrow 0$ as $n \rightarrow \infty$ for every j , it follows from (19) that

$$\int_{R_n} |\tilde{q}_n(w)| dw \wedge d\bar{w} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This, together with (14), implies

$$\int_{\mathbb{P}^1 \setminus \phi_{\tilde{\mu}_n}(Q_f)} |\tilde{q}_n(w)| dw \wedge d\bar{w} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This contradicts with the assumption (13) and completes the proof of the lemma. \square

By Propositions 4.1, 4.2 and Lemma 6.2, we have

Corollary 6.1. Let $b > 0$. Then there is a constant $0 < \delta < 1$ depending only on b such that

$$\|d\sigma_f|_\tau\| \leq \delta$$

for all $\tau \in T_{f,b}$.

Given any $[\mu_0] \in T_f$. Let $[\mu_n] = \sigma_f^n([\mu_0]) = [(f^*)^n \mu_0]$ for $n \geq 0$.

Lemma 6.3. Suppose that there exist a $b > 0$ and a point $[\mu_0] \in T_f$ such that $\{[\mu_n]\}_{n=0}^\infty \subset T_{f,b}$. Then σ_f has a unique fixed point in T_f .

Proof. From Corollary 6.1 and Lemma 3.1, it follows that $\{[\mu_n]\}_{n=0}^\infty$ is a Cauchy sequence. Since T_f is complete, $[\mu_n]$ converges to a limit point $[\mu]$ in T_f , that is,

$$\lim_{n \rightarrow \infty} [\mu_n] = [\mu].$$

It follows that $\sigma_f([\mu]) = [\mu]$. The uniqueness of the fixed point follows also from Corollary 6.1. \square

7. NO THURSTON OBSTRUCTION IMPLIES BOUNDED GEOMETRY

Lemma 7.1. Suppose that f has no Thurston obstructions. Then there is an integer $k > 0$ such that for every f -stable multi-curve $\Gamma = \{\gamma_i\}$ with $\gamma_i \subset S^2 \setminus Q_f$ and the associated linear transformation matrix A_Γ , we have

$$(24) \quad \max_j \sum_i b_{i,j} < 1/2$$

where $A_\Gamma^k = (b_{i,j})$.

Proof. Let $\Gamma = \{\gamma_i\}$ be a f -stable multi-curve with $\gamma_i \subset S^2 \setminus Q_f$. It is clear that the number of the elements in Γ has an upper bound which depends only on $\#(E)$. This implies that there can be only finitely many distinct A_Γ . The lemma follows. \square

Let $Z \subset S^2$ be a subset with $\#(Z) \geq 4$ and $\gamma \subset S^2 \setminus Z$ be a non-peripheral simple closed curve. For $[\mu] \in T_f$, define

$$w_Z(\gamma, [\mu]) = -\log \|\gamma\|_{\mu, Z}.$$

By using the same argument as in the proof of Proposition 7.2 of [5], we have

Lemma 7.2. *Let $Z \subset S^2$ be a subset with $\#(Z) \geq 4$ and $\gamma \subset S^2 \setminus Z$ be a non-peripheral simple closed curve. Then the function*

$$[\mu] \rightarrow w_Z(\gamma, [\mu]) : T_f \rightarrow \mathbb{R}$$

is Lipschitz with Lipschitz constant 2.

Recall that $E = P_1 \cup \cup_i \{a_i, b_i\}$. Let $[\mu] \in T_f$ and b be a real number. Define

$$\Gamma_\mu^b = \{\gamma \mid \gamma \text{ is a } (\mu, E)\text{-simple closed geodesic with } w_E(\gamma, [\mu]) \geq b\},$$

and

$$L_\mu = \{w_E(\gamma, [\mu]) \mid \gamma \text{ is a } (\mu, E)\text{-simple closed geodesic}\}.$$

Lemma 7.3. *There exists an $A > -\log \log \sqrt{2}$ such that for any $[\mu] \in T_f$ and any real numbers $a < b$, if*

1. $a > A$,
2. $b - a \geq \log d + 2d_T([\mu], [f^*\mu]) + 1$,
3. $[a, b] \cap L_\mu = \emptyset$,
4. $\Gamma_\mu^b \neq \emptyset$,

then Γ_μ^b is a f -stable multi-curve in $S^2 \setminus Q_f$.

Proof. Let $\gamma \in \Gamma_\mu^b$. By the first assertion of Lemma 5.6, γ is a non-peripheral and simple closed curve in $S^2 \setminus Q_f$ provided that A is big and thus $\|\gamma\|_{\mu, E}$ is small. By the second assertion of Lemma 5.6, we have

$$w_{Q_f}(\gamma, [\mu]) > w_E(\gamma, [\mu]) - 1$$

provided that A is big and thus $\|\gamma\|_{\mu, E}$ is small. Now suppose that γ' is a non-peripheral component of $f^{-1}(\gamma)$. Since f is a degree d branched covering map of the sphere, it follows that

$$w_{f^{-1}(Q_f)}(\gamma', [f^*\mu]) \geq w_{Q_f}(\gamma, [\mu]) - \log d.$$

Since $E \subset f^{-1}(Q_f)$, it follows that

$$w_E(\gamma', [f^*\mu]) > w_{f^{-1}(Q_f)}(\gamma', [f^*\mu]).$$

By Lemma 7.2, we have

$$w_E(\gamma', [\mu]) \geq w_E(\gamma', [f^*\mu]) - 2d_T([\mu], [f^*\mu]).$$

This implies that

$$w_E(\gamma, [\mu]) - w_E(\gamma', [\mu]) < \log d + 2d_T([\mu], [f^*\mu]) + 1 \leq b - a.$$

Since $w_E(\gamma, [\mu]) > b$ and $[a, b] \cap L_\mu = \emptyset$, it follows that $w_E(\gamma', [\mu]) > b$. This implies that γ' is homotopic to some element in Γ_μ^b . The Lemma follows. \square

Let $k \geq 0$ be the integer in Lemma 7.1. Let

$$(25) \quad P_2 = E \cup f^k(E) \cup \bigcup_{1 \leq j \leq k} f^j(\Omega_f).$$

Lemma 7.4. *There exists an $\epsilon_0 > 0$ which is independent of μ such that for any (μ, P_2) -simple closed geodesic η , if $\|\eta\|_{\mu, P_2} \leq \epsilon_0$, then there is a (μ, E) -simple closed geodesic γ such that η is homotopic to γ in $S^2 \setminus P_2$.*

Proof. Suppose η is not homotopic to any (μ, E) -simple closed geodesic in $S^2 \setminus P_2$. Then there is at least one holomorphic disk D_i , such that γ separate the points in $D_i \cap P_2$. Let $x, y \in D_i \cap P_2$ which are separated by γ . Let $z \in P_2 \setminus \overline{D_i}$. Let $\phi : S^2 \rightarrow \mathbb{P}^1$ be the homeomorphism which solves the Beltrami equation given by μ and which maps x, y , and z respectively to 0, 1, and the infinity. Then there are two cases.

In the first case, $\phi(\gamma)$ is contained in $\phi(\overline{D_i} \cup A_i)$. Note that A_i is the shielding ring attached to the outside of D_i . Then $\phi(\gamma)$ must enclose the ϕ -images of at least two points in $\overline{D_i} \cap P_2$. Since ϕ is univalent in $\overline{D_i} \cup A_i$, it follows from Koebe's distortion theorem that there exist $R > 0$ and $D > 0$ independent of μ such that $\phi(\gamma) \cap \{z | |z| \leq R\} \neq \emptyset$ and the Euclidean diameter of $\phi(\gamma)$ is greater than D . This, together with Koebe's distortion theorem, implies that the hyperbolic length of $\phi(\gamma)$ in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and thus $\|\gamma\|_{\mu, P_2}$, has a positive lower bound independent of μ .

In the second case, $\phi(\gamma)$ is not contained in $\phi(\overline{D_i} \cup A_i)$. Since $\phi(\gamma)$ separates $\phi(x)$ and $\phi(y)$, it follows that $\phi(\gamma)$ must cross through the annulus $\phi(A_i)$. By Koebe's distortion theorem, the annulus $\phi(A_i)$ has definite thickness. This again implies that there exist $R > 0$ and $D > 0$ independent of μ such that $\phi(\gamma) \cap \{z | |z| \leq R\} \neq \emptyset$ and the Euclidean diameter of $\phi(\gamma)$ is greater than D . Therefore, the hyperbolic length of $\phi(\gamma)$ in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and thus $\|\gamma\|_{\mu, P_2}$, has a positive lower bound independent of μ . The proof of the lemma is completed. \square

Note that

$$(26) \quad f^k : S^2 \setminus f^{-k}(P_2) \rightarrow S^2 \setminus P_2$$

is a covering map of degree d^k . Let $A > -\log \log \sqrt{2}$ be the constant in Lemma 7.3.

Lemma 7.5. *Let $B > A$. Then there exists a constant $M > 0$ depending only on the numbers $k, B, \#(E), \epsilon_0$, and the degree d of f , such that for any $[\mu] \in T_f$ and any real numbers $a < b$, if*

1. $A < a < B$,
2. $b - a \geq \log d + 2d_T([\mu], [f^*\mu]) + 1$,
3. $[a, b] \cap L_\mu = \emptyset$,
4. $\Gamma_\mu^b \neq \emptyset$,

then

$$\sum_{\gamma \in \Gamma_\mu^b} \frac{1}{\|\gamma\|_{\nu, E}} \leq \frac{1}{2} \sum_{\gamma \in \Gamma_\mu^b} \frac{1}{\|\gamma\|_{\mu, E}} + M,$$

where $\nu = (f^k)^*(\mu)$ and $k \geq 0$ is the integer in Lemma 7.1.

Proof. By Lemma 7.3, Γ_μ^b is a f -stable multi-curve in $S^2 \setminus Q_f$. For each $\gamma_j \in \Gamma_\mu^b$, let $\gamma_{i,j,\alpha}$ be any component of $f^{-k}(\gamma_j)$ homotopic to γ_i in $S^2 \setminus Q_f$. Then $\gamma_{i,j,\alpha}$ is also homotopic to γ_i in $S^2 \setminus E$.

Let $g = \phi_\mu \circ f^k \circ \phi_\nu^{-1}$. Then g is a rational map. It follows from (26) that

$$g : \mathbb{P}^1 \setminus \phi_\nu(f^{-k}(P_2)) \rightarrow \mathbb{P}^1 \setminus \phi_\mu(P_2)$$

is a holomorphic covering map, and therefore,

$$\|\gamma_{i,j,\alpha}\|_{\nu, f^{-k}(P_2)} = d_{i,j,\alpha} \|\gamma_j\|_{\mu, P_2}$$

where $d_{i,j,\alpha} \leq d^k$ is the degree of

$$f^k : \gamma_{i,j,\alpha} \rightarrow \gamma_j.$$

Thus

$$\sum_{\alpha} \frac{1}{\|\gamma_{i,j,\alpha}\|_{\nu, f^{-k}(P_2)}} = \left(\sum_{\alpha} \frac{1}{d_{i,j,\alpha}} \right) \frac{1}{\|\gamma_j\|_{\mu, P_2}} = b_{ij} \frac{1}{\|\gamma_j\|_{\mu, P_2}}$$

Since $E \subset P_2$ by (25), it follows that $\|\gamma_j\|_{\mu, P_2} > \|\gamma_j\|_{\mu, E}$, and therefore

$$\frac{1}{\|\gamma_j\|_{\mu, P_2}} < \frac{1}{\|\gamma_j\|_{\mu, E}}.$$

This implies

$$(27) \quad \sum_{\alpha} \frac{1}{\|\gamma_{i,j,\alpha}\|_{\nu, f^{-k}(P_2)}} < b_{ij} \frac{1}{\|\gamma_j\|_{\mu, E}}$$

Note that $E \subset f^{-k}(P_2)$ by (25). Let p denote the number of the points in $f^{-k}(P_2) \setminus E$. It follows from (25) that there is a constant

$$0 < C(k, d, \#(E)) < \infty$$

depending only on d , k , and $\#(E)$ such that $p \leq C(k, d, \#(E))$.

Now we claim that for any $(\nu, f^{-k}(P_2))$ -simple closed geodesic γ which is homotopic to γ_i in $S^2 \setminus E$, either γ is homotopic to some $\gamma_{i,j,\alpha}$ in $S^2 \setminus f^{-k}(P_2)$, or

$$\|\gamma\|_{\nu, f^{-k}(P_2)} > \min\{e^{-B}, \epsilon_0\}.$$

Let us prove the claim. In fact, if γ is not homotopic in $S^2 \setminus f^{-k}(P_2)$ to some $\gamma_{i,j,\alpha}$, then $f^k(\gamma)$ is a (μ, P_2) -simple closed geodesic which is not homotopic to any γ_j in $S^2 \setminus P_2$. There are two cases. In the first case, $f^k(\gamma)$ is homotopic in $S^2 \setminus P_2$ to some (μ, E) -simple closed geodesic ξ which does not belong to Γ_μ^b . By the assumption that $L_\mu \cap [a, b] = \emptyset$, we have

$$\|f^k(\gamma)\|_{\mu, P_2} > \|f^k(\gamma)\|_{\mu, E} = \|\xi\|_{\mu, E} > e^{-a} > e^{-B}.$$

In the second case, $f^k(\gamma)$ is not homotopic in $S^2 \setminus P_2$ to any (μ, E) -simple closed geodesic. By Lemma 7.4, we have

$$\|f^k(\gamma)\|_{\mu, P_2} > \epsilon_0.$$

We thus have

$$\|\gamma\|_{\nu, f^{-k}(P_2)} \geq \|f^k(\gamma)\|_{\mu, P_2} > \min\{e^{-B}, \epsilon_0\}.$$

Now from the left hand of the inequality given by (c) in Theorem 7.1 of [5], we have

$$\frac{1}{\|\gamma_i\|_{\nu, E}} \leq \sum_j \sum_{\alpha} \frac{1}{\|\gamma_{i,j,\alpha}\|_{\nu, f^{-k}(P_2)}} + \frac{2}{\pi} + \frac{C(k, d, \#(E)) + 1}{\min\{e^{-B}, \epsilon_0\}}.$$

Let

$$M' = \frac{2}{\pi} + \frac{C(k, d, \#(E)) + 1}{\min\{e^{-B}, \epsilon_0\}}.$$

Thus

$$\sum_{\gamma \in \Gamma_{\mu}^b} \frac{1}{\|\gamma\|_{\nu, E}} \leq \sum_i \sum_j \sum_{\alpha} \frac{1}{\|\gamma_{i,j,\alpha}\|_{\nu, f^{-k}(P_2)}} + KM'.$$

where K is the number of the curves in Γ which is bounded above by $\#(E) - 3$.
Let

$$M = (\#(E) - 3)M'.$$

By (27), we have

$$\sum_{\gamma \in \Gamma_{\mu}^b} \frac{1}{\|\gamma\|_{\nu, E}} \leq \sum_j \left(\sum_i b_{ij} \right) \frac{1}{\|\gamma_j\|_{\mu, E}} + M \leq \frac{1}{2} \sum_{\gamma \in \Gamma_{\mu}^b} \frac{1}{\|\gamma\|_{\mu, E}} + M.$$

This completes the proof of the Lemma. \square

The following is a technical lemma from Calculus.

Lemma 7.6. *Let $b_0 > 1$, $c_0, M_0 > 0$, and integer $m_0 > 1$ be given. Then for any sequence $\{x_n\}_{n=0}^{\infty}$ of positive numbers satisfying*

- (1) $x_0 \leq c_0$,
- (2) $x_{n+1}/x_n \leq b_0$,
- (3) if $x_n \geq M_0$, then $x_{n+m_0} \leq x_n$,

one has

$$x_n \leq \max\{b_0^{m_0-1}c_0, b_0^{m_0}M_0\}, \quad \forall n \geq 0.$$

Proof. Let $C = \max\{b_0^{m_0-1}c_0, b_0^{m_0}M_0\}$. It is sufficient to prove that

$$x_{i+lm_0} \leq C$$

for all $0 \leq i \leq m_0 - 1$ and $l \geq 0$. Take an arbitrary integer $0 \leq i \leq m_0 - 1$.

Let us prove that

$$x_{i+lm_0} \leq C$$

for all $l \geq 0$ by induction. For $l = 0$, we have

$$x_i \leq b_0^i x_0 \leq b_0^{m_0-1} c_0 \leq C.$$

Now assume that

$$(28) \quad x_{i+km_0} \leq C$$

for some integer $k \geq 0$. Let us prove that

$$x_{i+(k+1)m_0} \leq C.$$

In fact, there are two cases by assumption (28). In the first case, $x_{i+km_0} < M$. In this case, we have

$$x_{i+(k+1)m_0} \leq b_0^{m_0} x_{i+km_0} < b_0^{m_0} M \leq C.$$

In the second case, $x_{i+km_0} \geq M$. Then we have

$$x_{i+(k+1)m_0} \leq x_{i+km_0} \leq C.$$

This proves that $x_{i+lm_0} \leq C$ for all $l \geq 0$. Since this holds for any $0 \leq i \leq m_0 - 1$, the lemma follows. \square

Lemma 7.7. *If f has no Thurston obstructions, then for any $[\mu_0] \in T_f$, there exists a constant $b > 0$ such that for all $n \geq 1$,*

$$[\mu_n] \in T_{f,b},$$

where $\mu_n = (f^*)^n(\mu_0)$.

Proof. Since f is holomorphic on $\cup A_i$ and $f(\cup A_i) \subset \cup D_i$, it follows that for all $n \geq 1$, $\mu_n(z) = 0$ on $\cup A_i$. By Lemma 5.3, it is equivalent to prove that there is a uniform positive lower bound of the length of all the (μ_n, E) -simple closed geodesics.

Let $D = d_T([\mu_0], [\mu_1])$. Then by Lemma 3.1 and Corollary 6.1, we have

$$d_T([\mu_n], [\mu_{n+1}]) \leq D \quad \text{for all } n \geq 0.$$

Let $K = \#(E) - 3 \geq 1$ and $k \geq 1$ be the integer in Lemma 7.1. Let $l_0 \geq 1$ be the least integer such that

$$(29) \quad K < 2^{l_0-1}.$$

Now it is sufficient to prove that there exist positive constants $c_0, M_0 > 0$, $b_0 > 1$, and an integer $m_0 > 0$, such that the sequence

$$x_n = \max_{\gamma} \{ \|\gamma\|_{\mu_n, E}^{-1} \},$$

where max is taken over all the (μ_n, E) -simple closed geodesics, satisfies the three conditions in Lemma 7.6.

By Corollary 6.6 of [5], there are at most K (μ_n, E) -simple closed geodesics which has hyperbolic length less than $\log(\sqrt{2} + 1)$. This implies that we can have $c_0 > 0$ such that

$$x_0 \leq c_0.$$

It is the first condition in Lemma 7.6. From Lemma 7.2 we can take $b_0 = e^{2D}$.

Recall that we use d to denote the degree of f . Let $k_0 = \log d + 2D$ and $m_0 = kl_0$. Let

$$(30) \quad k_1 = k_0 + 4m_0D + 1.$$

In particular, $k_1 > \log d + 2D + 1$. Let $A > -\log \log(\sqrt{2} + 1)$ be the constant in Lemma 7.3. In Lemma 7.5, take

$$B = A + (K + 1)k_1$$

and let M denote the corresponding constant there. Let

$$M_0 = \max\{e^B, 2^{l_0+1}M\}.$$

It remains to prove that if $x_n > M_0$, then $x_{n+m_0} < x_n$. To see this, suppose that $x_n > M_0$. It follows that there is a (μ_n, E) -simple close geodesic such that $w_E(\gamma, [\mu_n]) > B$. Then by the choice of the numbers k_1 and B , and the fact that there are at most K (μ_n, E) -simple closed geodesics which have hyperbolic length less than $\log(\sqrt{2} + 1)$, one can take an interval $[a, b]$ such that

1. $A < a < b < B$,
2. $b - a = k_1$,
3. $[a, b] \cap L_{\mu_n} = \emptyset$.

It follows that $\Gamma_{\mu_n}^b \neq \emptyset$ and therefore is a f -stable multicurve by Lemma 7.3. Now for each $i = 0, 1, \dots, l_0$, let

$$[a_i, b_i] = [a + 2kiD, b - 2kiD].$$

By Lemma 7.2, the gap condition $b - a = k_1$, and (30), it follows that each family $\Gamma_{\mu_{n+ki}}^{b_i}$, $0 \leq i \leq l_0$, contains the same set of homotopy classes of simple closed curves as $\Gamma_{\mu_n}^b$. Let us simply denote each of them by Γ . Now for each $0 \leq i \leq l_0 - 1$, let $\mu = \mu_{n+ki}$ and $\nu = \mu_{n+k(i+1)}$, and let $[a_i, b_i]$ be the corresponding gap interval. Then the conditions in Lemma 7.5 are satisfied with the constants A and B given as above. By Lemma 7.5, we have

$$\sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_{n+k(i+1)}, E}} \leq \frac{1}{2} \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_{n+ki}, E}} + M$$

for $0 \leq i \leq l_0 - 1$. It follows from $m_0 = kl_0$ that

$$(31) \quad \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_{n+m_0}, E}} \leq \frac{1}{2^{l_0}} \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n, E}} + 2M.$$

Since

$$\sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n, E}} \geq x_n > M_0 \geq 2^{l_0+1}M,$$

it follows that

$$(32) \quad M < \frac{1}{2^{l_0+1}} \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n, E}}.$$

From (31) and (32), we have

$$(33) \quad \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_{n+m_0}, E}} < \frac{1}{2^{l_0-1}} \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n, E}}.$$

Since the number of the elements in Γ is at most K , it follows that

$$\sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n, E}} \leq Kx_n.$$

From (29) and (33), we have

$$x_{n+m_0} \leq \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_{n+m_0}, E}} < \frac{1}{2^{l_0-1}} \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n, E}} \leq \frac{K}{2^{l_0-1}} x_n < x_n.$$

□

The Main Theorem now follows from Lemmas 4.3, 6.3, and 7.7.

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